International Journal of Theoretical Physics, Vol. 28, No. 3, 1989

Modification of the Superstring Action and the Exceptional Jordan Algebra

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Received April 22, 1988

An attempt is made to modify the superstring action in order to incorporate exceptional groups as internal symmetry groups. This is achieved by allowing the string variables to take on values in the exceptional Jordan algebra. This results in an exceptional quantum mechanical space which provides the string with an exceptional symmetry group in a fundamentally different way from the prescription of Chan and Paton.

1. INTRODUCTION

Green and Schwarz (1984) have singled out $SO(32)$ and $E_8 \times E_8$ as gauge groups which admit an anomaly-free string theory. However, as is well known, the exceptional groups have been excluded from type 1 superstrings. Let us briefly recall why this is so. The internal group is introduced by associating "quark" charges with the ends of the string à la Chan and Paton (1969). One associates group generators λ_i with the ith external string state and introduces the group factor

$$\mathrm{tr}(\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_m}) \tag{1}$$

into the superstring amplitudes. Consistency with the factorization property of string amplitudes (Schwarz, 1982) then dictates that

$$\lambda \equiv \lambda_1\lambda_2\cdots\lambda_m - \lambda_m\lambda_{m-1}\cdots\lambda_1 \tag{2}$$

closes for all m. Marcus and Sagnotti (1982) have shown that equation (2) does not close for exceptional groups.

In this paper, we investigate an alternative approach to the incorporation of internal symmetry. Ultimately, a theory is sought whereby the exceptional group can appear as an internal symmetry of the theory. We

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0020-7748/89/0300-0263$06.00/0

Table I

$\mathbb{A}/J$	$\mathfrak{M}_3^1$	$\mathfrak{M}_3^2$	$\mathfrak{M}_3^4$	$\mathfrak{M}_3^8$
$\mathbb{R}$	$SO(3)$	$SU(3)$	$Sp(6)$	F_4
$\mathbb{C}$	$SU(3)$	$SU(3)\times SU(3)$	$SU(6)$	E_6
$\mathbb{H}$	$Sp(6)$	$SU(6)$	$SO(12)$	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

propose to utilize the result that the exceptional groups arise as automorphism groups of exceptional algebras, and that these algebras define a quantum theory. Our idea is to modify the first quantized superstring by allowing the string variables to take on values in the exceptional Jordan algebra (Jordan, 1933; Jordan *et al.*, 1934; Segal, 1947; Sherman, 1956). In a previous paper (Foot and Joshi, 1987a) we attempted to incorporate the exceptional Jordan algebra into the bosonic string. In that note we were exploring a possible connection between the space-time dimension $d=26$ and the dimensionality of the exceptional Jordan algebra $\mathfrak{M}_3^8$. Here our motivations are slightly different.

The reason for this inclusion of exceptional Jordan algebras in the superstring is because the exceptional groups are related to the Jordan algebras. The situation here is summarized nicely by the Freudenthal–Tits magic square (Table I). The Lie algebras in the square are given by

$$L = \mathrm{Der}\,\mathbb{A} + A_0 \times J_0 + \mathrm{Der}(J) \tag{3}$$

where Der $\mathbb{A}$ and Der J are the derivation algebras of the division algebra $\mathbb{A}$ and the Jordan algebra J, i.e., they are the Lie algebras associated with the automorphism group of the algebras $\mathbb{A}$ and J. Here A_0 and J_0 are the imaginary and traceless elements of the division algebra $\mathbb{A}$ and J, respectively.

In Section 2 we briefly review the Jordan formulation of quantum mechanics. In Section 3 we modify the superstring action by allowing the string variables to take values in the exceptional Jordan algebra $\mathfrak{M}_3^8$. As a consequence the string states include a Jordan matrix which transforms under the group F_4. We have in effect attached the finite-dimensional exceptional quantum mechanical space to the space of string states. Finally, in Section 4 we discuss our results and make some concluding remarks.

2. JORDAN FORMULATION OF QUANTUM MECHANICS

Gürsey (1975) has led the revival of interest in Jordan algebras, which has been followed by several authors (Truini and Biedenharn, 1981; Nambu, 1973). Jordan algebra can be defined axiomatically by the rule

$$A \times B = B \times A \tag{4}$$

and the property

$$(A, B, A^2) = 0 \tag{5}$$

where the associator $(\cdot, \cdot, \cdot)$ is defined by

$$(A, B, C) = (A \times B) \times C - A \times (B \times C) \tag{6}$$

Every Jordan algebra (with one important exception) is equivalent to an algebra $\mathfrak{M}$, containing real matrices with the (Jordan) product

$$X \circ Y \equiv 1/2(XY + YX) \tag{7}$$

where XY denotes ordinary matrix multiplication between X and Y. This algebra is called a special Jordan algebra. The one exception is the exceptional Jordan algebra $\mathfrak{M}_3^8$, the elements of which are the 3×3 Hermitian octonionic matrices

$$M = \begin{bmatrix} \alpha & a & \bar{b} \\ \bar{a} & \beta & c \\ b & \bar{c} & \gamma \end{bmatrix} \tag{8}$$

where α, β and γ are real and a, b, and c are arbitrary octonions.

In the Jordan formulation of quantum mechanics, states are represented by Hermitian projection operators

$$P_\alpha^+ = P_\alpha \tag{9}$$

In the special Jordan algebra one makes the association

$$P_\alpha \equiv |\alpha\rangle\langle\alpha| \tag{10}$$

and thus an explicit connection with the usual bra-ket Hilbert space is make. However, due to the nonassociativity of the octonions, this connection cannot be made for the exceptional Jordan matrices.

The transition probability is defined by

$$T_{\alpha\beta} = \mathrm{Tr}\, P_\alpha \circ P_\beta \tag{11}$$

Note that the (unobservable) probability amplitude cannot be expressed in the Jordan formalism. Also note that all products are Jordan products; thus, the Jordan algebra is manifestly commutative. In fact, using special Jordan algebra, the noncommutative but associative algebra of quantum mechanics can *equivalently* be replaced by the commutative but nonassociative Jordan algebra.

The transformation law for P_α is given by

$$P_\alpha = P_\alpha + 1/1!(h_1, P_\alpha, h_2) + 1/2!(h_1, (h_1, P_\alpha, h_2), h_2) + \cdots \tag{12}$$

For the algebra $\mathfrak{M}_3^8$, P_α are the 3×3 Hermitian octonionic matrices; h_1 and h_2 are traceless Jordan matrices. The group of such transformations (automorphisms) is F_4.

Let us now consider the scattering formalism. If the initial state is P_{in}, and after scattering is P_{out}, then in ordinary quantum mechanics they are related by

$$P_{\text{out}} = SP_{\text{in}}S^{-1} \tag{13}$$

The scattering matrix S defines an automorphism of the algebra. Hence in the exceptional case we have

$$P_{\text{out}} = P_{\text{in}} + 1/1!(h_1, P_{\text{in}}, h_2) + 1/2!(h_1, (h_1, P_{\text{in}}, h_2), h_2) + \cdots \tag{14}$$

Therefore the transition probability is given by

$$\begin{aligned} T_{fi} &= \operatorname{Tr} P_f \circ P_{\text{out}} \\ &= \operatorname{Tr} P_f \circ P_{\text{in}} + \operatorname{Tr} P_f \circ (h_1, P_{\text{in}}, h_2) + \cdots \end{aligned} \tag{15}$$

The exceptional Jordan quantum mechanics is *inequivalent* to the usual quantum mechanics. Indeed, this exceptional quantum mechanics cannot be interpreted within the usual framework of quantum mechanics. For instance, one cannot define a Hilbert space. We now utilize the exceptional Jordan quantum mechanics to construct a new string action which can accommodate the automorphism group F_4 as a symmetry group of the spectrum of states of the string.

3. EXCEPTIONAL SUPERSTRING

The conventional action for the superstring in light-cone gauge is

$$S = \int d\sigma\, d\tau - 1/4\pi\alpha'\, \partial_\alpha X^i\, \partial^\alpha X^i + i/4\pi \bar{S}\gamma^- \rho \cdot \partial S \tag{16}$$

Throughout this section we follow the notation of Schwarz (1982). To incorporate the $\mathfrak{M}_3^8$ matrix structure, we simply introduce the direct product

$$\mathbf{X}^i = X^i \otimes \mathbb{1}, \qquad \mathbf{S} = S \otimes \mathbb{1} \tag{17}$$

where $\mathbb{1}$ is the identity element of the exceptional Jordan algebra $\mathfrak{M}_3^8$ (it can be represented by a 3×3 unit matrix).

The product of two elements in our direct product space is defined by

$$\mathbf{A} \cdot \mathbf{B} = (AB) \otimes (J' \circ J) \tag{18}$$

where $\mathbf{A}=A\otimes J'$, $\mathbf{B}=B\otimes J$, and $J, J'\in\mathfrak{M}_3^8$, i.e., we have normal multiplication between A and B and Jordan multiplication between the $\mathfrak{M}_3^8$ part.

We now consider the following string action:

$$S'=\int d\sigma\, d\tau -1/4\pi\alpha'\,\partial_\alpha\mathbf{X}^i\,\partial^\alpha\mathbf{X}^i - i/4\pi\bar{\mathbf{S}}\gamma^-\rho\cdot\partial\mathbf{S}$$

$$\Rightarrow S'=S\otimes\mathbb{1}. \tag{19}$$

Note that the S' is invariant under the automorphism group F_4, as it is clear that the automorphism group must preserve the identity [i.e., $\delta\mathbb{1}=0$, as can easily be checked using equation (12)].

The modification of the string action S by allowing the string variables to take on values in $\mathfrak{M}_3^8$ leads to a new theory. The new action S' has all the invariances of S plus the F_4 group invariance. Because of the simplicity of this modification of the string action, the subsequent analysis of S' is straightforward. In fact, the reader may wonder whether we have done anything at all! We shall see we have.

The equations of motion are

$$\begin{aligned}(\partial_\sigma^2-\partial_\tau^2)\mathbf{X}^i&=0\\(\partial_\sigma+\partial_\tau)\mathbf{S}^{1a}&=0\\(\partial_\tau-\partial_\sigma)\mathbf{S}^{2a}&=0\end{aligned} \tag{20}$$

If we restrict our attention to open strings, then the appropriate boundary conditions are

$$\begin{aligned}\partial_\sigma\mathbf{X}^i|_{\sigma=0}=\partial_\sigma\mathbf{X}^i|_{\sigma=\pi}&=0\\\mathbf{S}^{1a}(0,\tau)&=\mathbf{S}^{2a}(0,\tau)\\\mathbf{S}^{1a}(\pi,\tau)&=\mathbf{S}^{2a}(\pi,\tau)\end{aligned} \tag{21}$$

The solution of equations (20) with boundary conditions (21) is

$$\begin{aligned}\mathbf{X}^i(\sigma,\tau)&=\mathbf{X}^i+\mathbf{p}^i\tau+i\sum_{n\neq0}1/n\,\boldsymbol{\alpha}_n^i\cos n\sigma\, e^{-in\tau}\\\mathbf{S}^{1a}(\sigma,\tau)&=\sum_{n=-\infty}^{\infty}\mathbf{S}_n^a\,e^{-in(\tau-\sigma)}\\\mathbf{S}^{2a}(\sigma,\tau)&=\sum_{n=-\infty}^{\infty}\mathbf{S}_n^a\,e^{-in(\tau+\sigma)}\end{aligned} \tag{22}$$

and canonical quantization gives

$$\begin{aligned}[\mathbf{X}^i,\mathbf{P}^j]&=i\eta^{ij}\otimes\mathbb{1}\\[\boldsymbol{\alpha}_m^i,\boldsymbol{\alpha}_n^j]&=m\delta_{m+n,0}\delta^{ij}\otimes\mathbb{1}\\\{\mathbf{S}_m^a,\bar{\mathbf{S}}_n^b\}&=(\gamma^+h)^{ab}\delta_{m+n,0}\otimes\mathbb{1}\\[\boldsymbol{\alpha}_m^i,\mathbf{S}_n^a]&=0\end{aligned} \tag{23}$$

The $(\text{mass})^2$ operator is

$$\alpha'(\text{mass})^2 = N = \sum_{n=1}^{\infty} \boldsymbol{\alpha}_{-n}^{i} \cdot \boldsymbol{\alpha}_{n}^{i} + n/2\, \bar{\mathbf{S}}_{-n} \cdot \gamma^{-} \mathbf{S}_{n}$$

$$= \sum_{n=1}^{\infty} \alpha_{-n}^{i} \alpha_{n}^{i} \otimes \mathbb{1} + n/2\, \bar{S}_{-n} \gamma^{-} S_{n} \otimes \mathbb{1} \tag{24}$$

This seems all very trivial; however, the states of this string are not. The point is, that by enlarging the string variables by incorporating the tensor product with the exceptional quantum mechanical space we can satisfy the usual requirements, i.e., Lorentz algebra, Virasoro algebra, etc. However, the states of the string can be nontrivial and we can associate a 3×3 Jordan matrix with the states of the string. The general state is of the form

$$|\alpha\rangle\langle\alpha| \otimes \Pi \tag{25}$$

where $|\alpha\rangle$ is the usual Fock space, i.e., the space on which the operators α_n^i act. And Π is any element of $\mathfrak{M}_3^8$ satisfying

$$\Pi^2 = \Pi \qquad \text{and} \qquad \operatorname{Tr} \Pi = 1 \tag{26}$$

These matrices may be written as

$$\Pi = \begin{pmatrix} a \\ w_1 \\ w_2 \end{pmatrix} (a \ \ w_1 \ \ w_2) \tag{27}$$

where a is real and w_i are octonions which satisfy the condition

$$a^2 + \bar{w}_1 w_1 + \bar{w}_2 w_2 = 1 \tag{28}$$

Hence we arrive at the following mass spectrum. The massless ground state is

$$\begin{aligned} &\text{Bose} \qquad |i\rangle\langle i| \otimes \Pi, \qquad i = 1, \ldots, 8 \\ &\text{Fermi} \qquad |a\rangle\langle a| \otimes \Pi, \qquad a = 1, \ldots, 8 \end{aligned} \tag{29}$$

where $|a\rangle = i/8(\gamma_i S_0)^a |i\rangle$. The first excited state is

$$\begin{aligned} &\text{Bose} \qquad \boldsymbol{\alpha}_{-i}^{i} \cdot (|j\rangle\langle j| \otimes \Pi \cdot \boldsymbol{\alpha}_{1}^{i}) \\ &\qquad\qquad \mathbf{S}_{-1}^{a} \cdot (|b\rangle\langle b| \otimes \Pi \cdot \mathbf{S}_{1}^{a}) \\ &\text{Fermi} \qquad \boldsymbol{\alpha}_{-1}^{i} \cdot (|a\rangle\langle a| \otimes \Pi \cdot \boldsymbol{\alpha}_{1}^{i}) \\ &\qquad\qquad \mathbf{S}_{-1}^{a} \cdot (|i\rangle\langle i| \otimes \Pi \cdot \mathbf{S}_{1}^{a}) \end{aligned} \tag{30}$$

etc., for higher mass states.

Note that the states now have an internal group structure. The states are in the fundamental representation of F_4. Let us now investigate the scattering formalism. The transition probability is given by

$$T_{fi} = \mathrm{Tr}(P_f \cdot P_{\mathrm{out}}) \tag{31}$$

where

$$\begin{aligned} P_f &= |f\rangle\langle f| \otimes \Pi_f \\ P_{\mathrm{out}} &= |\mathrm{out}\rangle\langle\mathrm{out}| \otimes \Pi_{\mathrm{out}} \end{aligned} \tag{32}$$

and

$$\begin{aligned} &|\mathrm{out}\rangle\langle\mathrm{out}| = S|i\rangle\langle i|S^+ \\ &\Pi_{\mathrm{out}} = \Pi_{\mathrm{in}} + 1/1!(h_1, \Pi_{\mathrm{in}}, h_2) + 1/2!(h_1, (h_1, \Pi_{\mathrm{in}}, h_2), h_2) + \cdots \end{aligned} \tag{33}$$

Hence

$$T_{fi} = \mathrm{Tr}(\Pi_f \circ \Pi_{\mathrm{in}} + \Pi_f \circ (h_1, \Pi_{\mathrm{in}}, h_2) + \cdots)|\langle f|S|i\rangle|^2 \tag{34}$$

The transition probability has a charge-space (i.e., internal symmetry)-independent part $|\langle f|S|i\rangle|^2$ multiplied with a pure charge space part. Clearly, the internal symmetry is incorporated in a fundamentally different manner from that of Chan and Paton, or indeed, in conventional gauge theories. Observe that this scattering formalism is only defined at the level of a first quantized quantum mechanical one-particle theory; it is not clear how to extend it to a many-particle scattering formalism. Thus, the form of the transition probability for N particles has not been determined.

4. DISCUSSION AND CONCLUDING REMARKS

By a simple modification of the superstring action within the context of Jordan quantum mechanics we have established a superstring with an F_4 internal symmetry, the states of the string belonging to the fundamental representation of F_4. Furthermore, this symmetry group has been incorporated in a way which is profoundly different from the procedure of Chan and Paton.

There are of course several problems with this theory. First, the states of the string are restricted by equation (26). Not every state in the fundamental representation is present in the string spectrum. Second, we have the wrong group (F_4 instead of $E_8 \times E_8$). Third, if we investigate closed strings in this model, i.e., we replace the boundary conditions (21) with the closed string boundary conditions

$$\begin{aligned} \mathbf{X}^i(0, \tau) &= \mathbf{X}^i(\pi, \tau) \\ \mathbf{S}^{Aa}(0, \tau) &= \mathbf{S}^{Aa}(\pi, \tau) \end{aligned} \tag{35}$$

then proceeding as in Section 3, we would find that the closed string states also have this F_4 group structure. Finally, we have not considered string interactions.

If we try to generalize this theory in order to include the other exceptional groups, then we run into some difficulties. The problem is that the unit matrix $\mathbb{1}$ is no longer invariant under the action of the automorphism group. For example, consider the infinitesimal E_6 transformation (Gürsey, 1975)

$$\delta F = (H_1, F, H_2) + iH_3 F \tag{36}$$

where F is a complex octonionic Jordan matrix, and H_i are traceless Hermitian 3×3 matrices over real octonions. Clearly

$$\delta \mathbb{1} = iH_3 \neq 0 \tag{37}$$

An equivalent statement of the problem is that the Freudenthal algebra (this is the algebra which has the exceptional group E_6 as the automorphism group) does not contain an identity element. Thus, extending this theory to the other exceptional groups is nontrivial.

Finally, we note that the exceptional Jordan algebra has been related to the superstring in several different ways. The structure group of the exceptional Jordan algebra contains the Lorentz group in 10 dimensions, with the vector and spinor being represented as elements of the exceptional Jordan algebra (Foot and Joshi, 1987b). Also, the algebra of vertex operators may be related to the exceptional Jordan algebra (Goddard *et al.*, 1987; Corrigan and Hollowood, 1988; Ferreira *et al.*, 1988; Günaydin and Hyun, 1988). Clearly, then, there are many indications that Jordan and related algebras may have relevance to superstring theory.

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